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An hybrid initial value method for singularly perturbed delay differential equations with interior layers and weak boundary layer

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KEYWORDS

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Abstract In this paper, an hybrid initial value method on Shishkin mesh is suggested to solve singularly perturbed boundary value problem for second order ordinary delay differential equation with discontinuous convection coefficient and source term. In this method, the original problem of solving the second order differential equation is reduced to solving four first order differential equations. Among the four first order differential equations, three of them are singularly perturbed differential equations without delay and other one is a regular differential equation with a delay term. The singularly perturbed differential equations are solved by the second order hybrid finite difference schemes, whereas the delay differential equation is solved by the improved Euler method. An error estimate is derived by using the supremum norm and it is of almost second order convergence. Numerical results are provided to illustrate the theoretical results.

1. Introduction

In this paper a singularly perturbed boundary value problem for second order ordinary delay differential equation with discontinuous convection coefficient and source term is considered. It is important to develop suitable numerical methods to solve singularly perturbed differential equations with a delay, whose accuracy does not depend on the parameter ε, that is, the methods are uniformly convergent with respect to the parameter, for more details one may refer to [1–6]. In the past, only very few people had worked in the area numerical methods to Singularly Perturbed Delay Differential Equation (SPDDE). But in recent years, there has been growing interest in this area, for more details refer [3–8] and the references therein. In fact, a B-spline collocation method and a finite difference method for small delay problems have been suggested in [4,5] respectively. The authors in [6,8] have suggested respectively, an uniformly valid finite difference method and an initial value method for convection diffusion problems with smooth data whereas in [7] the authors have suggested an initial value technique for singularly perturbed problem with non-smooth data. The author in [3] has suggested a uniformly valid finite difference method with linear interpolation on Shishkin mesh for second order ordinary delay differential equation with discontinuous convection coefficient and it is

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of first order convergence. In the present paper, an hybrid initial value method on Shishkin mesh is suggested for the problem considered in [3]. The order of convergence of the present method is almost second order.

The present paper is organized as follows. In the present section, the problem under study with discontinuous data is stated. Existence and stability results of the solution of the problem are established in Section 2. Some analytical results for the problem considered in this paper are given in Section 3. The present numerical method is described in Section 4 and an error estimate is derived in Section 5. Section 6 presents numerical results.

Throughout the paper, we assume that C, C_1 denote generic positive constants independent of the singular perturbation parameter $\varepsilon$and the discretization parameter N of the discrete problem and I_N denotes $\{0, 1, \ldots, N\}$. The supremum norm is used for studying the convergence of the numerical solution to the exact solution of a singular perturbation problem: $\|u\|_\Omega = \sup_{x\in\Omega}|u(x)|$.

Motivated by the works of [3,9–11], we consider the following BVP for SPDDE.

Find $u \in Y = C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^*)$ such that

$$\begin{cases} -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x-1) = f(x), & x \in \Omega^*, \\ u(x) = \phi(x), \; x \in [-1, 0], \; u(2) = l, \end{cases} \tag{1.1}$$

$a(x) = \begin{cases} a_1(x), x\in[0,1], \\ a_2(x), x\in(1,2], \end{cases}$ $f(x) = \begin{cases} f_1(x), x\in[0,1], \\ f_2(x), x\in(1,2], \end{cases}$ $a_1(1-) \neq a_2(1+), f_1(1-) \neq f_2(1+), \alpha^* \geqslant a_1(x) \geqslant \alpha_1 > \alpha > 0, -\alpha^* \leqslant a_2(x) \leqslant -\alpha_2 < -\alpha < 0, \; \alpha < \min\{\alpha_1, \alpha_2\}, \; \beta_0 \leqslant b(x) \leqslant \beta_1 < 0, \alpha + 2\beta_0 \geqslant \eta_0 > 0$ where $0 < \varepsilon \ll 1, a, f$ are sufficiently smooth and bounded in Ω^*. The function b is a sufficiently smooth function on $\overline{\Omega}, \Omega = (0,2), \overline{\Omega} = [0,2], \Omega^* = \Omega^- \cup \Omega^+, \Omega^- = (0,1), \Omega^+ = (1,2)$ and ϕ is smooth on $[-1,0]$. The above problem is equivalent to

$$Pu(x) := \begin{cases} -\varepsilon u''(x) + a_1(x)u'(x) = f_1(x) - b(x)\phi(x-1), & x \in \Omega^-, \\ -\varepsilon u''(x) + a_2(x)u'(x) + b(x)u(x-1) = f_2(x), & x \in \Omega^+, \end{cases} \tag{1.2}$$

$$u(0) = \phi(0), \quad u(1-) = u(1+), \quad u'(1-) = u'(1+), \quad u(2) = l,$$

where $u(1-)$ and $u(1+)$ denote the left and right limits of u at $x = 1$, respectively.

2. Existence and stability results

Theorem 2.1. *The problem (1.1) has a solution $u \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^*)$.*

Proof. See [3]. □

Theorem 2.2. *Let $w \in C^0(\overline{\Omega}) \cap C^2(\Omega^*)$ be any function satisfying $w(0) \geqslant 0, w(2) \geqslant 0, Pw(x) \geqslant 0, \; \forall x \in \Omega^*$ and $w'(1+) - w'(1-) = [w'](1) \leqslant 0$. Then $w(x) \geqslant 0, \; \forall x \in \overline{\Omega}$.*

Proof. See [3]. □

Corollary 2.3. *For any $u \in Y$ we have $|u(x)| \leqslant C\max\{|u(0)|, |u(2)|, \sup_{\xi\in\Omega^*}|Pu(\xi)|\}, \; \forall \; x \in \overline{\Omega}$.*

Proof. See [3]. □

3. Analytical results

In this section, an asymptotic expansion approximation for the solution of the problem (1.1) is constructed by using the fundamental idea of WKB method [12] and the procedure adopted in [7,8]. Further, the bounds of the derivatives of the solution are given.

Theorem 3.1. *Let u be the solution of the problem (1.2). Then*

$$|u^{(k)}(x)| \leqslant C \begin{cases} 1 + \varepsilon^{-k} \exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right), & x \in \Omega^-, \\ 1 + \varepsilon^{-k} \exp\left(\frac{-\alpha(x-1)}{\varepsilon}\right) + \varepsilon^{-k+1} \exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right), \\ \quad x \in \Omega^+, \; k = 0, 1, 2, 3. \end{cases}$$

Proof. See [3]. □

Note: The above theorem says that, the problem (1.1) has a weak boundary layer at $x = 2$ and strong interior layers at $x = 1$.

An asymptotic expansion approximation to the solution is given in (3.5).

Let $u_0 \in C^0(\Omega^* \cup \{0, 2\}) \cap C^1(\Omega^*)$ be the solution of the reduced problem of (1.1) given by

$$\begin{cases} a(x)u_0'(x) + b(x)u_0(x-1) = f(x), & x \in \Omega^*, \\ u_0(x) = \phi(x), \; x \in [-1, 0], \; u_0(2) = l. \end{cases} \tag{3.1}$$

Further, let $w_1(x) = \exp\left(-\int_x^1 \frac{a_1(s)}{\varepsilon} ds\right), \; x \in [0, 1], w_2(x) = \exp\left(\int_1^x \frac{a_2(s)}{\varepsilon} ds\right), \; x \in [1, 2]$ and $w_3(x) = \exp\left(\int_x^2 \frac{a_2(s)}{\varepsilon} ds\right)$, $x \in [1, 2]$ be the solutions of the following problems, respectively:

$$L_1 w_1(x) := \varepsilon w_1'(x) - a_1(x)w_1(x) = 0, \quad x \in [0, 1), \quad w_1(1) = 1, \tag{3.2}$$

$$L_2 w_2(x) := \varepsilon w_2'(x) - a_2(x)w_2(x) = 0, \quad x \in (1, 2], \quad w_2(1) = 1, \tag{3.3}$$

$$L_3 w_3(x) := \varepsilon w_3'(x) + a_2(x)w_3(x) = 0, \quad x \in [1, 2), \quad w_3(2) = 1. \tag{3.4}$$

The following theorem gives the estimate of the solutions and their derivatives of the aforesaid problems (3.2)–(3.4).

Theorem 3.2. *Let w_1, w_2 and w_3 be the solutions of the problems (3.2)–(3.4), respectively. Then, for $k = 0, 1, 2, 3$ we have,*

$$|w_1^{(k)}(x)| \leqslant C\varepsilon^{-k} \exp(-\alpha(1-x)/\varepsilon), \quad x \in [0, 1],$$
$$|w_2^{(k)}(x)| \leqslant C\varepsilon^{-k} \exp(-\alpha(x-1)/\varepsilon), \quad x \in [1, 2],$$
$$|w_3^{(k)}(x)| \leqslant C\varepsilon^{-k} \exp(-\alpha(2-x)/\varepsilon), \quad x \in [1, 2].$$

Proof. See [2]. □

An asymptotic expansion approximation to the solution of the original problem (1.1) is given by

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$$u_{as}(x)=\begin{cases}u_0(x)+k_1[w_1(x)-w_1(0)], & x\in[0,1]\\ u_0(x)+k_2[w_2(x)-w_2(2)w_3(x)], & x\in[1,2].\end{cases}\tag{3.5}$$

Here the constants k_1 and k_2 are to be determined such that $u_{as}\in Y$. In fact the constants k_1 and k_2 are given by

$$k_2=\frac{a_1(1-)[u_0(1+)-u_0(1-)]-\varepsilon[u_0'(1+)-u_0'(1-)][1-w_1(0)]}{a_2(1+)[1+w_2(2)w_3(1)][1-w_1(0)]-a_1(1-)[1-w_2(2)w_3(1)]},\tag{3.6}$$

$$k_1=\frac{[u_0(1+)-u_0(1-)]+k_2[1-w_2(2)w_3(1)]}{[1-w_1(0)]}.\tag{3.7}$$

It is easy to see that k_1 and k_2 are bounded constants.

Theorem 3.3. *Let u be the solution of (1.1) and u_{as} be its asymptotic expansion approximation given by (3.5). Then, $\|u-u_{as}\|_{\overline{\Omega}}\leqslant C\varepsilon$.*

Proof. Consider the barrier function $\varphi^{\pm}(x)=C_1\varepsilon\psi(x)\pm[u(x)-u_{as}(x)],\ x\in[0,2]$ where

$$\psi(x)=\begin{cases}s(x)+\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right)-\varepsilon\exp\left(\frac{-\alpha}{\varepsilon}\right), & x\in[0,1]\\ s(x)+\eta(x)-\varepsilon\exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right), & x\in[1,2],\end{cases}$$

$$\eta(x)=\begin{cases}0, & x\in[0,1),\\ \exp\left(\frac{-\alpha(x-1)}{\varepsilon}\right), & x\in[1,2]\end{cases}\quad\text{and}\quad s(x)=\begin{cases}\frac{3}{2}+\frac{x}{2}, & x\in[0,1],\\ 3-x, & x\in[1,2].\end{cases}$$

It is easy to see that $\varphi^{\pm}(0)\geqslant 0$ and $\varphi^{\pm}(2)\geqslant 0$.

Let $x\in\Omega^-$. Then

$$\begin{aligned}P\varphi^{\pm}(x)&=C_1\left[\varepsilon a_1(x)s'(x)+\alpha(a_1(x)-\alpha)\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right)\right]\\&\quad\pm P(u(x)-u_{as}(x))\\&\geqslant CC_1\left[\varepsilon+\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right)\right]\pm P(u(x)-u_{as}(x)).\end{aligned}$$

Note that $P(u(x)-u_{as}(x))=k_1w_1(x)a_1'(x)+\varepsilon u_0''(x)\geqslant -C(\varepsilon+\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right))$. Therefore $P\varphi^{\pm}(x)\geqslant 0,\ x\in\Omega^-$ for a suitable $C_1>0$. Similarly one can prove that $P\varphi^{\pm}(x)\geqslant 0,\ x\in\Omega^+$. By using the Theorem 2.2, we can prove the desired result. □

4. Numerical methods

In this section, mesh selection strategy namely piecewise uniform mesh (Shishkin mesh) is explained also an improved Euler scheme for the problem (3.1) and hybrid finite difference schemes on Shishkin mesh for the problems (3.2)–(3.4) are described.

4.1. Mesh selection strategy

Since the BVP (1.1) exhibits strong interior layers at $x=1$ and a weak boundary layer at $x=2$, we choose a piecewise uniform Shishkin mesh on $[0,2]$. For this we divide the interval $[0,2]$ into six subintervals, namely $\Omega_1=[0,\tau],\Omega_2=[\tau,1-\tau]$, $\Omega_3=[1-\tau,1]$, $\Omega_4=[1,1+\tau]$ $\Omega_5=[1+\tau,2-\tau]$, $\Omega_6=[2-\tau,2]$, where $\tau=\min\left\{0.25,\frac{2\varepsilon\ln N}{\alpha}\right\}$. Let $h_1=8N^{-1}\tau,h_2=4N^{-1}(1-2\tau)$. The mesh $\overline{\Omega}^N=\{x_i:i\in I_N\}$ is defined by $x_0=0.0$, $x_i=x_0+ih_1,i=1(1)N/8$, $x_{i+N/8}=x_{N/8}+ih_2,i=1(1)N/4,x_{i+3N/8}=x_{3N/8}+ih_1$, $i=1(1)N/4$, $x_{i+5N/8}=x_{5N/8}+ih_2,i=1(1)N/4,x_{i+7N/8}=x_{7N/8}+ih_1,i=1(1)N/8$.

4.2. Difference scheme for reduced problem (3.1)

On $\overline{\Omega}^N$, we define the improved Euler scheme for the problem (3.1):

$$\frac{U_{0_i}-U_{0_{i-1}}}{h(i)}=\frac{1}{2}\left[\frac{f_{1_i}}{a_{1_i}}+\frac{f_{1_{i-1}}}{a_{1_{i-1}}}-\frac{b_i}{a_{1_i}}\phi(x_i-1)-\frac{b_{i-1}}{a_{1_{i-1}}}\phi(x_{i-1}-1)\right],$$
$$i=1(1)\frac{N}{2},\quad U_{0_0}=\phi(x_0),\tag{4.1}$$

$$\frac{U_{0_{i+1}}-U_{0_i}}{h(i+1)}=\frac{1}{2}\left[\frac{f_{2_i}}{a_{2_i}}-\frac{b_i}{a_{2_i}}U_{0_{i-N/2}}+\frac{f_{2_{i+1}}}{a_{2_{i+1}}}-\frac{b_{i+1}}{a_{2_{i+1}}}U_{0_{i+1-N/2}}\right],$$
$$i=N/2+1(1)N-1,\quad U_{0_N}=l,\tag{4.2}$$

where $h(i)=x_i-x_{i-1},a_{1_i}=a_1(x_i)$ and $f_{1_i}=f_1(x_i)$.

Theorem 4.1. *Let u_0 be the solution of the problem (3.1) and its numerical solution is given by (4.1) and (4.2). Then $\|u_0-U_0\|_{\bar{\Omega}^N}\leqslant C\bar{h}^2$, where $\bar{h}=\max\{h_1,h_2\}$.*

Proof. See [13]. □

4.3. Difference schemes for (3.2)–(3.4)

On $\overline{\Omega}^N$, we define the following schemes for the problems defined in (3.2)–(3.4):

$$L_1^NW_{1_i}=\begin{cases}\varepsilon\frac{W_{1_{i+1}}-W_{1_i}}{h(i+1)}-a_{1_i}W_{1_i}=0,\ i=0,1(1)\frac{3N}{8},\\ \varepsilon\frac{W_{1_{i+1}}-W_{1_i}}{h(i+1)}-a_{1_{i+1/2}}\frac{W_{1_i}+W_{1_{i+1}}}{2}=0, i=\frac{3N}{8}+1(1)\frac{N}{2}-1,\end{cases}\quad W_{1_{N/2}}=1\tag{4.3}$$

$$L_2^NW_{2_i}=\begin{cases}\varepsilon\frac{W_{2_i}-W_{2_{i-1}}}{h(i)}-a_{2_{i-1/2}}\frac{W_{2_{i-1}}+W_{2_i}}{2}=0,\ i=\frac{N}{2}+1(1)\frac{5N}{8},\\ \varepsilon\frac{W_{2_i}-W_{2_{i-1}}}{h(i)}-a_{2_i}W_{2_i}=0,\ i=\frac{5N}{8}+1(1)N,\end{cases}\quad W_{2_{N/2}}=1\tag{4.4}$$

$$L_3^NW_{3_i}=\begin{cases}\varepsilon\frac{W_{3_{i+1}}-W_{3_i}}{h(i+1)}+a_{2_i}W_{2_i}=0,\ i=\frac{N}{2}(1)\frac{7N}{8},\\ \varepsilon\frac{W_{3_{i+1}}-W_{3_i}}{h(i+1)}+a_{2_{i+1/2}}\frac{W_{2_i}+W_{2_{i+1}}}{2}=0,\ i=\frac{7N}{8}+1(1)N-1,\end{cases}\quad W_{3_N}=1,\tag{4.5}$$

where $a_{1_{i+1/2}}=a_1\left(\frac{x_i+x_{i+1}}{2}\right)$ and $a_{2_{i-1/2}}=a_2\left(\frac{x_i+x_{i-1}}{2}\right)$.

Remark: In [14], the condition $\varepsilon\leqslant CN^{-1}$ was assumed to prove the order of the convergence $O(N^{-2}\log^2N)$. Without the condition we can prove the result. To illustrate that, we consider the hybrid finite difference scheme (4.3). The matrix associated with the aforesaid scheme (4.3) is given by $A=(a_{i,j})$, where

$$a_{1,1}=\frac{\varepsilon}{h(1)}+a_{1_0},\quad a_{1,2}=-\frac{\varepsilon}{h(1)},$$
$$a_{i+1,i+1}=\frac{\varepsilon}{h(i+1)}+a_{1_i},\quad a_{i+1,i+2}=-\frac{\varepsilon}{h(i+1)},\quad i=1(1)\frac{3N}{8},$$
$$a_{i+1,i+1}=\frac{\varepsilon}{h(i+1)}+\frac{a_{1_{i+1}}}{2},\quad a_{i+1,i+2}=\frac{a_{1_{i+1}}}{2}-\frac{\varepsilon}{h(i+1)},$$
$$i=\frac{3N}{8}+1(1)\frac{N}{2}-1,a_{\frac{N}{2}+1,\frac{N}{2}+1}=1$$

and the rest of the entries are zeros.

Table 1 Numerical results for the problem stated in Example 6.1.

	N (number of mesh points)						
	16	32	64	128	256	512	1024
D^N	4.9030e−2	1.8614e−2	5.9389e−3	2.0545e−3	6.4475e−4	1.9713e−4	5.7391e−5
p^N	1.3973	1.6481	1.5314	1.6720	1.7096	1.7802	–

Table 2 Numerical results for the problem stated in Example 6.2.

	N (number of mesh points)						
	16	32	64	128	256	512	1024
	Numerical results given in [3, Table 1]						
D^N	7.2967e−2	4.7273e−2	3.9152e−2	2.7566e−2	1.8534e−2	1.1663e−2	6.9885e−3
p^N	6.2625e−1	2.7192e−1	5.0620e−1	5.7270e−1	6.6825e−1	7.3887e−1	–
	Present method						
D^N	1.6876e−2	5.7049e−3	1.9683e−3	6.3403e−4	1.9627e−4	5.8499e−5	1.7414e−5
p^N	1.5647	1.5352	1.6344	1.6917	1.7463	1.7482	–

Table 3 Numerical results for the problem stated in Example 6.3.

	N (number of mesh points)						
	16	32	64	128	256	512	1024
	Numerical results given in [3, Table 2]						
D^N	4.7088e−2	2.8800e−2	2.4542e−2	1.6827e−2	1.1337e−2	7.1177e−3	4.2869e−3
p^N	7.0929e−1	2.3083e−1	5.4448e−1	5.6974e−1	6.7152e−1	7.3149e−1	–
	Present method						
D^N	1.2291e−2	4.3446e−3	1.4308e−3	4.8136e−4	1.4964e−4	4.4724e−5	1.3303e−5
p^N	1.5003	1.6024	1.5716	1.6856	1.7424	1.7493	–

Suppose $8\frac{\alpha^*}{\alpha}\log N \leqslant N$, then $\frac{a_{1_{i+1}}}{2} - \frac{\varepsilon}{h(i+1)} \leq 0$. That is, the off diagonal entries of A are non-positive and main diagonal entries are positive. It is easy to see that, for the vector $K = (1,1,\ldots,1) \in \mathbb{R}^{N/2+1}, AK^T > 0$. Then, A^{-1} exists and $A^{-1} \geqslant 0$ [15]. Therefore A is monotone [15,16]. Then we get the following discrete maximum principle.

Theorem 4.2. *Let W_{1_i} be any mesh function satisfying $W_{1_{N/2}} \geqslant 0$ and $L_1^N W_{1_i} \geqslant 0, i = 0(1)N/2-1$, then $W_i \geqslant 0, i = 0(1)N/2$.*

An immediate consequence of this theorem is the following stability result.

Theorem 4.3. *Let W_{1_i} be any mesh function, then*

$$|W_{1_i}| \leqslant C\ \max\{|W_{1_{N/2}}|,\quad \max_j\{|L_1^N W_{1_j}|\}\},\quad i = 0(1)N/2.$$

Proof. Using the barrier function $\varphi_i^{\pm} = CC_1 \pm W_{1_i},\ i = 0(1)N/2$ where $C_1 = \max\{|W_{1_{N/2}}|, \max_j\{|L_1^N W_{1_j}|\}\}$ one can prove the result. □

The following theorem gives an error estimate for the above scheme.

Theorem 4.4. *Let w_1 be the solution of the terminal value problem (3.2) and W_{1_i} be its numerical solution defined by (4.3). Then, $|w_{1_i} - W_{1_i}| \leqslant CN^{-2}\log^2 N,\ i = 0(1)N/2$.*

Proof. Expanding w_1 using the Taylor's expansion about $\frac{x_i+x_{i+1}}{2}, x_{i+1}$ and x_i respectively we get

$$w_1(x) = w_{1_{i+1/2}} + \left(x - \frac{x_{i+1}+x_i}{2}\right)w'_{1_{i+1/2}} + \frac{w''_{1_{i+1/2}}}{2!}\left(x - \frac{x_{i+1}+x_i}{2}\right)^2 + \cdots$$

$$w_1(x) = w_{1_{i+1}} + (x - x_{i+1})w'_{1_{i+1}} + \frac{w''_{1_{i+1}}}{2!}(x - x_{i+1})^2 + \frac{w'''_{1_{i+1}}}{3!}(x - x_{i+1})^3 + \cdots$$

$$w_1(x) = w_{1_i} + (x - x_i)w'_{1_i} + \frac{w''_{1_i}}{2!}(x - x_i)^2 + \frac{w'''_{1_i}}{3!}(x - x_i)^3 + \cdots,$$

where $w_{1_i} = w_1(x_i)$. From the aforesaid equations, we have

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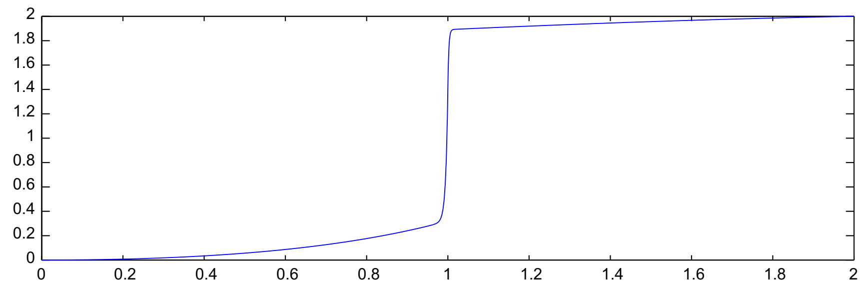

Figure 1 Numerical solution of the problem stated in Example 6.1.

$$w_1(x_{i+1}) = w_{1_{i+1/2}} + \left(\frac{h(i+1)}{2}\right) w'_{1_{i+1/2}} + \frac{w''_{1_{i+1/2}}}{2!}\left(\frac{h(i+1)}{2}\right)^2 + \frac{w'''_{1_{i+1/2}}}{3!}\left(\frac{h(i+1)}{2}\right)^3 + \cdots \quad (4.6)$$

$$w_1(x_i) = w_{1_{i+1/2}} - \left(\frac{h(i+1)}{2}\right) w'_{1_{i+1/2}} + \frac{w''_{1_{i+1/2}}}{2!}\left(\frac{h(i+1)}{2}\right)^2 - \frac{w'''_{1_{i+1/2}}}{3!}\left(\frac{h(i+1)}{2}\right)^3 + \cdots, \quad (4.7)$$

where $h(i) = x_i - x_{i-1}$. From the aforesaid Eqs. (4.6) and (4.7), we have

$$\left|\frac{w_1(x_{i+1}) - w_1(x_i)}{h(i+1)} - w'_{1_{i+1/2}}\right| \leqslant C\left|\frac{1}{24}(h(i+1))^2 w'''_{1_{i+1/2}}\right|, \quad (4.8)$$

$$\left|\frac{w_1(x_{i+1}) + w_1(x_i)}{2} - w_{1_{i+1/2}}\right| \leqslant C\left|\frac{1}{8}(h(i+1))^2 w''_{1_{i+1/2}}\right|. \quad (4.9)$$

Consider a mesh function

$$\varphi_i^{\pm} = \begin{cases} C_1 N^{-2} \pm (w_1(x_i) - W_{1_i}), & i = 0(1)3N/8, \\ C_1 N^{-2}\log^2 N \pm (w_1(x_i) - W_{1_i}), & i = 3N/8 + 1(1)N/2. \end{cases}$$

It is easy to see that $\varphi^{\pm}_{N/2} \geqslant 0$ for a suitable choice of C_1. Further

$$L_1^N\varphi_i^{\pm} = \begin{cases} C_1 L_1^N(N^{-2}) \pm L_1^N(w_1(x_i) - W_{1_i}), & i = 0(1)3N/8, \\ C_1 L_1^N(N^{-2}\log^2 N) \pm L_1^N(w_1(x_i) - W_{1_i}), & i = 3N/8 + 1(1)N/2. \end{cases}$$

For $i = 0(1)3N/8$,

$$L_1^N(w_1(x_i) - W_{1_i}) = \varepsilon\left(\frac{w_{1_{i+1}} - w_{1_i}}{h(i+1)} - w'_{1_i}\right)$$

$$\begin{aligned}\left|L_1^N(w_1(x_i) - W_{1_i})\right| &\leqslant \frac{\varepsilon}{h(i+1)}\int_{x_i}^{x_{i+1}} |x - x_i||w_1''(x)|dx \\ &\leqslant \varepsilon\int_{x_i}^{x_{i+1}} |w_1''(x)|dx \\ &\leqslant C\varepsilon^{-1}\int_{x_i}^{x_{i+1}} \exp(-\alpha x/\varepsilon)dx \\ &\leqslant C\exp(-\alpha x_i/\varepsilon)(1 - \exp(-\alpha h_1/\varepsilon))\end{aligned}$$

$$\left|L_1^N(w_1(x_i) - W_{1_i})\right| \leqslant C\exp(-\alpha\tau/\varepsilon) \leqslant CN^{-2}.$$

For $i = 3N/8 + 1(1)N/2$,

$$\begin{aligned}L_1^N(w_1(x_i) - W_{1_i}) &= L_1^N w_1(x_i) - (\varepsilon w'_{1_{i+1/2}} - a_{1_{i+1/2}} w_{1_{i+1/2}}) \\ &= \varepsilon\left(\frac{w_1(x_{i+1}) - w_1(x_i)}{h(i+1)} - w'_{1_{i+1/2}}\right) \\ &\quad - a_{1_{i+1/2}}\left(\frac{w_1(x_{i+1}) + w_1(x_i)}{2} - w_{1_{i+1/2}}\right)\end{aligned}$$

From Eqs. (4.8) and (4.9),

$$\begin{aligned}\left|L_1^N(w_1(x_i) - W_{1_i})\right| &\leqslant \varepsilon\left|\frac{w_1(x_{i+1}) - w_1(x_i)}{h(i+1)} - w'_{1_{i+1/2}}\right| \\ &\quad + a_{1_{i+1/2}}\left|\frac{w_1(x_{i+1}) + w_1(x_i)}{2} - w_{1_{i+1/2}}\right| \\ &\leqslant C\left(\varepsilon\left|\frac{1}{24}h(i+1)^2 w'''_{1_{i+1/2}}\right| + \left|\frac{1}{8}h(i+1)^2 w''_{1_{i+1/2}}\right|\right) \\ &\leqslant Ch_1^2\varepsilon^{-2}\exp(-\alpha(x_{i+1} + x_i)/2\varepsilon) \\ &\leqslant C(\tau N^{-1})^2\varepsilon^{-2} = C\left(\frac{2}{\alpha}N^{-1}\varepsilon\log N\right)^2\varepsilon^{-2} \\ &\leqslant CN^{-2}\log^2 N, \quad i = 3N/8 + 1(1)N/2.\end{aligned}$$

Hence $L_1^N\varphi_i^{\pm} \geqslant 0$, $i = 0(1)N/2 - 1$ for a suitable choice of C_1. Then by the Theorem 4.2, $\varphi_i^{\pm} \geqslant 0$, $i = 0(1)N/2$. That is, $|w_1(x_i) - W_{1_i}| \leqslant CN^{-2}(\log^2 N)$, $i = 0(1)N/2$. □

Similarly we can prove the following results.

Theorem 4.5. *Let w_2 and w_3 be the solutions of the problems (3.3) and (3.4), respectively. Further, let $W_2$and W_3 be the numerical solutions of (3.3) and (3.4), respectively defined by (4.4) and (4.5). Then $\|w_2 - W_2\|_{[1,2]\cap\bar{\Omega}^N} \leqslant CN^{-2}\log^2 N$ and $\|w_3 - W_3\|_{[1,2]\cap\bar{\Omega}^N} \leqslant CN^{-2}\log^2 N$.*

4.4. A numerical solution to the BVP (1.1)

A numerical solution of the original problem is given by

$$U_i = \begin{cases} U_{0_i} + k_1[W_{1_i} - w_1(0)], & i = 1(1)N/2 \\ U_{0_i} + k_2[W_{2_i} - w_2(2)W_{3_i}], & i = N/2 + 1(1)N \end{cases} \quad (4.10)$$

where $U_{0_i}, W_{1_i}, W_{2_i}$, and W_{3_i} are the numerical solutions of problems (3.1)–(3.4) and the constants k_1, k_2 are given in (3.6) and (3.7).

5. Error analysis

In this section we derive an error estimate for the numerical solution obtained by (4.10) for the problem (1.1).

Theorem 5.1. *Let $u(x)$ be the solution of the problem (1.1), further let U_i be its numerical solution defined by (4.10). Then* $\|u - U\|_{\bar{\Omega}^N} \leqslant C(\varepsilon + N^{-2}\log^2 N)$.

Proof. From Theorems 3.3, 4.1, 4.4 and 4.5, we have $\|u - u_{as}\|_{\overline{\Omega}} \leqslant C\varepsilon$, $\|w_1 - W_1\|_{[0,1]\cap\bar{\Omega}^N} \leqslant CN^{-2}\log^2 N$, $\|w_2 - W_2\|_{[1,2]\cap\bar{\Omega}^N} \leqslant CN^{-2}\log^2 N$ and $\|w_3 - W_3\|_{[1,2]\cap\bar{\Omega}^N} \leqslant CN^{-2}\log^2 N$ and $\|u_0 - U_0\|_{\overline{\Omega}^N} \leqslant CN^{-2}$. Then

$$|u(x_i) - U_i| \leqslant |u(x_i) - u_{as}(x_i)| + |u_{as}(x_i) - U_i|,\ i \in I_N$$
$$\leqslant \begin{cases} |u(x_i) - u_{as}(x_i)| + |u_0(x_i) - U_{0_i}| + |k_1||w_1(x_i) - W_{1_i}|, \\ \quad i = 0(1)N/2 \\ |u(x_i) - u_{as}(x_i)| + |u_0(x_i) - U_{0_i}| + |k_2|[|w_2 - W_{2_i}| \\ \quad + |w_2(2)||w_3(x_i) - W_{3_i}|],\ i = N/2+1(1)N \end{cases} \leqslant C\begin{cases} \varepsilon + N^{-2} + N^{-2}\log^2 N,\ i = 0(1)N/2 \\ \varepsilon + N^{-2} + N^{-2}\log^2 N,\ i = N/2+1(1)N \end{cases} \leqslant C\begin{cases} \varepsilon + N^{-2}\log^2 N,\ i = 0(1)N/2 \\ \varepsilon + N^{-2}\log^2 N,\ i = N/2+1(1)N. \end{cases}$$

That is, $|u(x_i) - U_i| \leqslant C(\varepsilon + N^{-2}\log^2 N),\ i \in I_N$. □

6. Numerical examples

In this section, three examples are given to illustrate the numerical method discussed in this paper. We use the double mesh principle to estimate the error and compute the experiment rate of convergence in our computed solution for all problems. For this we put $D_\varepsilon^M = \max_{0\leqslant i\leqslant M}|U_i^M - U_{2i}^{2M}|$, where U_i^M and U_{2i}^{2M} are the i^{th} components of the numerical solutions on meshes of M and $2M$ points respectively. We compute the uniform error and rate of convergence as $D^M = \max_\varepsilon D_\varepsilon^M$ and $p^M = \log_2\left(\frac{D^M}{D^{2M}}\right)$. For the following examples the numerical results are presented for the values of perturbation parameter $\varepsilon \in \{2^{-27},\ 2^{-26}, \ldots, 2^{-6}\}$.

Example 6.1.

$$\begin{cases} -\varepsilon u''(x) + 3u'(x) - u(x-1) = \exp(x^2),\ x \in \Omega^- \\ -\varepsilon u''(x) - (4 + \exp(x))u'(x) - u(x-1) = -1,\ x \in \Omega^+ \\ u(x) = x,\ x \in [-1, 0],\ u(2) = 2. \end{cases}$$

Table 1 presents the values of D^N and p^N for this problem.

Example 6.2. [3, Example 8.1]

$$\begin{cases} -\varepsilon u''(x) + 3u'(x) - u(x-1) = 0,\ x \in \Omega^- \\ -\varepsilon u''(x) - 4u'(x) - u(x-1) = 0,\ x \in \Omega^+ \\ u(x) = 1,\ x \in [-1, 0],\ u(2) = 2. \end{cases}$$

Table 2 presents the values of D^N and $p^N$for this problem.

Example 6.3. [3, Example 8.2]

$$\begin{cases} -\varepsilon u''(x) + (3 + x^2)u'(x) - u(x-1) = 1,\ x \in \Omega^- \\ -\varepsilon u''(x) - (4 + x)u'(x) - u(x-1) = -1,\ x \in \Omega^+ \\ u(x) = 1,\ x \in [-1, 0],\ u(2) = 2. \end{cases}$$

Table 3 presents the values of D^N and p^N for this problem.

7. Discussion

A BVP for one type of SPDDEs is considered. To obtain an approximate solution for this type of problems, an initial value method on Shishkin mesh is presented. The method is shown to be of order $O(\varepsilon + N^{-2}\log^2 N)$. This very much reflected in the numerical results (Tables 1–3). A comparison of the maximum error given in [3] and the maximum error of the present paper is given in Tables 2 and 3. From Tables 2 and 3, it is easy to see that, the maximum error D^N of the present method is smaller than the maximum error D^N given in [3]. Fig. 1 exhibits that, the problem stated in Example 6.1 has strong interior layers at $x = 1$ and weak boundary layer at $x = 2$. The condition $\varepsilon \leqslant CN^{-1}$ was assumed in [14] to prove almost second order convergence. But without this condition one can prove the second order convergence for homogeneous first order singularly perturbed ordinary differential equations, see Theorem 4.4.

References

[1] Farrell PA, Hegarty AF, Miller JJH, O'Riordan E, Shishkin GI. Robust computational techniques for boundary layers. Boca Raton: Chapman Hall/CRC; 2000.
[2] Miller JJH, O'Riordan E, Shishkin GI. Fitted numerical methods for singular perturbation problems: error estimates in the maximum norm for linear problems in one and two dimensions. Singapore: World Scientific Publishing Co. Pte. Ltd; 2012.
[3] Subburayan V. A parameter uniform numerical method for singularly perturbed delay problems with discontinuous convection coefficient. Arab J Math Sci 2015. http://dx.doi.org/10.1016/j.ajmsc.2015.07.001.
[4] Kadalbajoo MK, Kumar D. Fitted mesh B-spline collocation method for singularly perturbed differential equations with small delay. Appl Math Comput 2008;204:90–8.
[5] Mohapatra J, Natesan S. Uniform convergence analysis of finite difference scheme for singularly perturbed delay differential equation on an adaptively generated grid. Numer Math Theory Methods Appl 2010;3(1):1–22.
[6] Amiraliyev GM, Cimen C. Numerical method for a singularly perturbed convection–diffusion problem with delay. Appl Math Comput 2010;216:2351–9.

[7] Subburayan V, Ramanujam N. Asymptotic initial value technique for singularly perturbed convection diffusion delay problems with boundary and weak interior layers. Appl Math Lett 2012;25 (12):2272–8.
[8] Subburayan V, Ramanujam N. An initial value technique for singularly perturbed convection–diffusion problems with a delay. J Optim Theory Appl 2013;158(1):234–50.
[9] Farrell PA, Hegarty AF, Miller JJH, O'Riordan E, Shishkin GI. Global maximum norm parameter-uniform numerical method for a singularly perturbed convection-diffusion problems with discontinuous convection coefficient. Math Comput Model 2004;40:1375–92.
[10] Zhongdi Cen. A hybrid difference scheme for a singularly perturbed convection-diffusion problem with discontinuous convection coefficient. Appl Math Comput 2005;169:689–99.
[11] Lange CG, Miura RM. Singularly perturbation analysis of boundary-value problems for differential-difference equations. Siam J Appl Math 1982;42(3):502–30.
[12] Bender Carl M, Orszag Steven A. Advanced mathematical methods for scientists and engineers. Singapore: McGraw-Hill Book Co.; 1984.
[13] Süli Endre, Mayers David F. An introduction to numerical analysis. Cambridge University Press; 2003.
[14] Cen Zhongdi. A hybrid finite difference scheme for a class of singularly perturbed delay differential equations. Neural Parallel Sci Comput 2008;16:303–8.
[15] George Poole, Thomas Boullion. A survey on M-matrices. SIAM Rev 1974;16(4):419–27.
[16] Jain MK. Numerical solution of differential equations. New Delhi, India: New Age International Publishers; 1984.

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